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WEIGHTED C^k ESTIMATES FOR A CLASS OF INTEGRAL OPERATORS ON NON-SMOOTH DOMAINS

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Abstract. We apply integral representations for $(0,q)$-forms, $q \geq 1$, on non-smooth strictly pseudoconvex domains, the Henkin-Leiterer domains, to derive weighted C^k estimates for a given $(0,q)$-form, f, in terms of C^k norms of $\bar{\partial} f$, and $\bar{\partial}^* f$. The weights are powers of the gradient of the defining function of the domain.

1. Introduction

Let X be an n-dimensional complex manifold, equipped with a Hermitian metric, and $D \subset\subset X$ a strictly pseudoconvex domain with defining function r. Here we do not assume the non-vanishing of the gradient, dr, thus allowing for the possibility of singularities in the boundary, ∂D of D. We refer to such domains as Henkin-Leiterer domains, as they were first systematically studied by Henkin and Leiterer in [2].

We shall make the additional assumtion that r is a Morse function.

Let $\gamma = |\partial r|$. In [1] the author established an integral representation of the form

Theorem 1.1. *There exist integral operators $\tilde{T}_q : L^2_{(0,q+1)}(D) \rightarrow L^2_{(0,q)}(D)$ $0 \leq q < n = \dim X$ such that for $f \in L^2_{(0,q)} \cap Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ one has*

$$\gamma^3 f = \tilde{T}_q \bar{\partial} f + \tilde{T}^*_{q-1} \bar{\partial}^* f + \text{ error terms} \qquad \text{for } q \geq 1.$$

Theorem 1.1 is valid under the assumption we are working with the Levi metric. With local coordinates denoted by $\zeta_1, \ldots, \zeta_n$, we define a Levi metric in a neighborhood of ∂D by

$$ds^2 = \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j, \partial \bar{\zeta}_k}(\zeta).$$

A Levi metric on X is a Hermitian metric which is a Levi metric in a neighborhood of ∂D. From what follows we will be working with X equipped with a Levi metric.

The author then used properties of the operators in the representation to establish the estimates

Theorem 1.2. *For $f \in L^2_{0,q}(D) \cap Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$, $q \geq 1$,*

$$\|\gamma^{3(n+1)} f\|_{L^\infty} \lesssim \|\gamma^2 \bar{\partial} f\|_\infty + \|\gamma^2 \bar{\partial}^* f\|_\infty + \|f\|_2.$$

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In this paper, we examine the operators in the integral representation, derive more detailed properties of such operators under differentiation, and use the properties to establish C^k estimates. Our main theorem is

Theorem 1.3. *Let $f \in L^2_{0,q}(D) \cap Dom(\bar\partial) \cap Dom(\bar\partial^*)$, $q \geq 1$, and $\alpha < 1/4$. Then for $N(k)$ large enough we have*

$$\|\gamma^{N(k)} f\|_{C^{k+\alpha}} \lesssim \|\gamma^{k+2}\bar\partial f\|_{C^k} + \|\gamma^{k+2}\bar\partial^* f\|_{C^k} + \|f\|_2.$$

We show we may take any $N(k) > 3(n+6) + 8k$.

Our results are consistent with those obtained by Lieb and Range in the case of smooth strictly pseudoconvex domains [4], where we may take $\gamma = 1$. In, [4], an estimate as in Theorem 1.3 with $\gamma = 1$ and $\alpha < 1/2$ was given.

In a separate paper we look establish C^k estimates for $f \in L^2(D) \cap \mathrm{Dom}(\bar\partial)$, as the functions used in the construction of the integral kernels in the case $q = 0$ differ from those in the case $q \geq 1$.

One of the difficulties in working on non-smooth domains is the problem of the choice of frame of vector fields with which to work. In the case of smooth domains a special boundary chart is used in which $\omega^n = \partial r$ is part of an orthonormal frame of $(1,0)$-forms. When ∂r is allowed to vanish, the frame needs to be modified. We get around this difficulty by defining a $(1,0)$-form, ω^n by $\partial r = \gamma\omega^n$. In the dual frame of vector fields we are then faced with factors of γ in the expressions of the vector fields with respect to local coordinates, and we deal with these terms by multiplying our vector fields by a factor of γ. This ensures that when vector fields are commuted, there are no error terms which blow up at the singularity.

We organize our paper as follows. In Section 2 we define the types of operators which make up the integral representation established in [1]. Section 3 contains the most essential properties used to obtain our results. In Section 3 we consider the properties of our integral operators under differentiation. Lastly, in Section 4 we apply the properties from Section 3 to obtain our C^k estimates.

The author extends thanks to Ingo Lieb with whom he shared many fruitful discussions over the ideas presented here, and from whom he originally had the idea to extend results on smooth domains to Henkin-Leiterer domains.

2. Admissible operators

With local coordinates denoted by $\zeta_1, \ldots, \zeta_n$, we define a Levi metric in a neighborhood of ∂D by

$$ds^2 = \sum_{j,k} \frac{\partial^2 r}{\partial\zeta_j, \partial\bar\zeta_k}(\zeta) d\zeta_j d\bar\zeta_k.$$

A Levi metric on X is a Hermitian metric which is a Levi metric in a neighborhood of ∂D.

We thus equip X with a Levi metric and we take $\rho(x,y)$ to be a symmetric, smooth function on $X \times X$ which coincides with the geodesic distance in a neighborhood of the diagonal, Λ, and is positive outside of Λ.

For ease of notation, in what follows we will always work with local coordinates, ζ and z.

Since D is strictly pseudoconvex and r is a Morse function, we can take $r_\epsilon = r + \epsilon$ for epsilon small enough. Then r_ϵ will be defining functions for smooth, strictly

pseudoconvex D_ϵ. For such r_ϵ we have that all derivatives of r_ϵ are indpendent of ϵ. In particular, $\gamma_\epsilon(\zeta) = \gamma(\zeta)$ and $\rho_\epsilon(\zeta, z) = \rho(\zeta, z)$.

Let F be the Levi polynomial for D_ϵ:

$$F(\zeta, z) = \sum_{j=1}^{n} \frac{\partial r_\epsilon}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 r_\epsilon}{\partial \zeta_j \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k).$$

We note that $F(\zeta, z)$ is independent of ϵ since derivatives of r_ϵ are.

For ϵ small enough we can choose $\delta > 0$ and $\varepsilon > 0$ and a patching function $\varphi(\zeta, z)$, independent of ϵ, on $\mathbb{C}^n \times \mathbb{C}^n$ such that

$$\varphi(\zeta, z) = \begin{cases} 1 & \text{for } \rho^2(\zeta, z) \leq \frac{\varepsilon}{2} \\ 0 & \text{for } \rho^2(\zeta, z) \geq \frac{3}{4}\varepsilon, \end{cases}$$

and defining $S_\delta = \{\zeta : |r(\zeta)| < \delta\}$, $D_{-\delta} = \{\zeta : r(\zeta) < \delta\}$, and

$$\phi_\epsilon(\zeta, z) = \varphi(\zeta, z)(F_\epsilon(\zeta, z) - r_\epsilon(\zeta)) + (1 - \varphi(\zeta, z))\rho^2(\zeta, z),$$

we have the following

Lemma 2.1. *On $D_\epsilon \times D_\epsilon \bigcap S_\delta \times D_{-\delta}$,*

$$|\phi_\epsilon| \gtrsim |\langle \partial r_\epsilon(z), \zeta - z\rangle| + \rho^2(\zeta, z),$$

where the constants in the inequalities are independent of ϵ.

We at times have to be precise and keep track of factors of γ which occur in our integral kernels. We shall write $\mathcal{E}_{j,k}(\zeta, z)$ for those double forms on open sets $U \subset D \times D$ such that $\mathcal{E}_{j,k}$ is smooth on U and satisfies

$$\mathcal{E}_{j,k}(\zeta, z) \lesssim \xi_k(\zeta)|\zeta - z|^j, \tag{2.1}$$

where ξ_k is a smooth function in D with the property

$$|\gamma^\alpha D_\alpha \xi_k| \lesssim \gamma^k,$$

for D_α a differential operator of order α.

We shall write $\mathcal{E}_j$ for those double forms on open sets $U \subset D \times D$ such that $\mathcal{E}_j$ is smooth on U, can be extended smoothly to $\overline{D} \times \overline{D}$, and satisfies

$$\mathcal{E}_j(\zeta, z) \lesssim |\zeta - z|^j.$$

$\mathcal{E}^*_{j,k}$ will denote forms which can be written as $\mathcal{E}_{j,k}(z, \zeta)$.

For $N \geq 0$, we let R_N denote an N-fold product, or a sum of such products, of first derivatives of $r(z)$, with the notation $R_0 = 1$.

Here

$$P_\epsilon(\zeta, z) = \rho^2(\zeta, z) + \frac{r_\epsilon(\zeta)}{\gamma(\zeta)} \frac{r_\epsilon(z)}{\gamma(z)}.$$

Definition 2.2. A double differential form $\mathcal{A}^\epsilon(\zeta, z)$ on $\overline{D}_\epsilon \times \overline{D}_\epsilon$ is an *admissible* kernel, if it has the following properties:

i) $\mathcal{A}^\epsilon$ is smooth on $\overline{D}_\epsilon \times \overline{D}_\epsilon - \Lambda_\epsilon$

ii) For each point $(\zeta_0, \zeta_0) \in \Lambda_\epsilon$ there is a neighborhood $U \times U$ of (ζ_0, ζ_0) on which $\mathcal{A}^\epsilon$ or $\overline{\mathcal{A}}^\epsilon$ has the representation

$$R_N R^*_M \mathcal{E}_{j,\alpha} \mathcal{E}^*_{k,\beta} P_\epsilon^{-t_0} \phi_\epsilon^{t_1} \overline{\phi}_\epsilon^{t_2} \phi_\epsilon^{*t_3} \overline{\phi}_\epsilon^{*t_4} r_\epsilon^l r_\epsilon^{*m} \tag{2.2}$$

with $N, M, \alpha, \beta, j, k, t_0, \ldots, m$ integers and $j, k, t_0, l, m \geq 0$, $-t = t_1 + \cdots + t_4 \leq 0$, $N, M \geq 0$, and $N + \alpha, M + \beta \geq 0$.

The above representation is of *smooth type* s for

$$s = 2n + j + \min\{2, t - l - m\} - 2(t_0 + t - l - m).$$

We define the *type* of $\mathcal{A}^\epsilon(\zeta, z)$ to be

$$\tau = s - \max\{0, 2 - N - M - \alpha - \beta\}.$$

$\mathcal{A}^\epsilon$ has *smooth type* $\geq s$ if at each point (ζ_0, ζ_0) there is a representation (2.2) of smooth type $\geq s$. $\mathcal{A}^\epsilon$ has *type* $\geq \tau$ if at each point (ζ_0, ζ_0) there is a representation (2.2) of type $\geq \tau$. We shall also refer to the *double type* of an operator (τ, s) if the operator is of type τ and of smooth type s.

The definition of smooth type above is taken from [5]. Here and below $(r_\epsilon(x))^* = r_\epsilon(y)$, the $*$ having a similar meaning for other functions of one variable.

Let $\mathcal{A}_j^\epsilon$ be kernels of type j. We denote by $\mathcal{A}_j$ the pointwise limit as $\epsilon \to 0$ of $\mathcal{A}_j^\epsilon$ and define the double type of $\mathcal{A}_j$ to be the double type of the $\mathcal{A}_j^\epsilon$ of which it is a limit. We also denote by A_j^ϵ to be operators with kernels of the form $\mathcal{A}_j^\epsilon$. A_j will denote the operators with kernels $\mathcal{A}_j$. We use the notation $\mathcal{A}_{(j,k)}^\epsilon$ (resp. $\mathcal{A}_{(j,k)}$) to denote kernels of double type (j, k).

We let $\mathcal{E}_{j-2n}^i(\zeta, z)$ be a kernel of the form

$$\mathcal{E}_{j-2n}^i(\zeta, z) = \frac{\mathcal{E}_{m,0}(\zeta, z)}{\rho^{2k}(\zeta, z)} \qquad j \geq 1,$$

where $m - 2k \geq j - 2n$. We denote by E_{j-2n} the corresponding isotropic operator.

From [1], we have

Theorem 2.3. *For $f \in L^2_{(0,q)}(D) \cap Dom(\bar\partial) \cap Dom(\bar\partial^*)$, there exist integral operators T_q, S_q, and P_q such that*

$$\gamma(z)^3 f(z) = \gamma^* T_q \bar\partial\left(\gamma^2 f\right) + \gamma^* S_q \bar\partial^*\left(\gamma^2 f\right) + \gamma^* P_q\left(\gamma^2 f\right).$$

T_q, S_q, and P_q have the form

$$\begin{aligned} T_q &= E_{1-2n} + A_1 \\ S_q &= E_{1-2n} + A_1 \\ P_q &= \frac{1}{\gamma} A^\epsilon_{(-1,1)} + \frac{1}{\gamma^*} A^\epsilon_{(-1,1)} \end{aligned}$$

3. Estimates

We begin with estimates on the kernels of a certain type. In [1], we proved the

Proposition 3.1. *Let A_j be an operator of type j. Then*

$$A_j : L^p(D) \to L^s(D) \quad \frac{1}{s} > \frac{1}{p} - \frac{j}{2n+2}.$$

We describe what we shall call tangential derivatives on the Henkin-Leiterer domain, D. A non-vanishing vector field, T, in $\mathbb{R}^{2n}$ will be called tangential if $Tr = 0$ on $r = 0$. Near a boundary point, we choose a coordinate patch on which we have an orthogonal frame $\omega^1, \ldots, \omega^n$ of $(1,0)$-forms with $\partial r = \gamma\omega^n$. Let $L_1, \ldots, L_n$ denote the dual frame. $L_1, \ldots, L_{n-1}$, $\overline{L}_1, \ldots, \overline{L}_{n-1}$, and $Y = L_n - \overline{L}_n$ are tangential vector fields. $N = L_n + \overline{L}_n$ is a normal vector field. We say a given vector field X is a smooth tangential vector field if it is a tangential field and if near each boundary point X is a combination of such vector fields $L_1, \ldots, L_{n-1}$, $\overline{L}_1, \ldots, \overline{L}_{n-1}, Y$, and

rN with coefficients in $C^\infty(\overline{D})$. We make the important remark here that in the coordinate patch of a critical point, the smooth tangential vector fields are not smooth combinations of derivatives with respect to the coordinate system described in Lemma 3.9. In fact, they are combinations of derivatives with respect to the coordinates of Lemma 3.9 with coefficients only in $C^0(\overline{D})$ due to factors of γ which occur in the denominators of such coefficients. In general a k^{th} order derivative of such coefficients is in $\mathcal{E}_{0,-k}$. Thus, when integrating by parts, special attention has to be paid to these non-smooth terms.

Definition 3.2. We say an operator with kernel, $\mathcal{A}$, is of *commutator* type j if $\mathcal{A}$ is of type j, and if in the representation of $\mathcal{A}$ in (2.2) we have $t_1t_3 \geq 0$, $t_2t_4 \geq 0$, and $(t_1+t_3)(t_2+t_4) \leq 0$.

Definition 3.3. Let W be a smooth tangential vector field on $\overline{D}$. We call W *allowable* if for all $\zeta \in \partial D$

$$W^\zeta \in T^{1,0}_\zeta(\partial D) \oplus T^{0,1}_\zeta(\partial D).$$

The following theorem is obtained by a modification of Theorem 2.20 in [4] (see also [3]). The new details which come about from the fact that here we do not assume $|\partial r| \neq 0$ require careful consideration and so we go through the calculations below.

Theorem 3.4. *Let A_1 be an admissible operator of commutator type ≥ 1 and X a smooth tangential vector field. Then*

$$\gamma^* X^z A_1 = -A_1 \tilde{X}^\zeta \gamma + A_1^{(0)} + \sum_{\nu=1}^{l} A_1^{(\nu)} W_\nu^\zeta \gamma,$$

where $\tilde{X}$ is the adjoint of X, the W_ν are allowable vector fields, and the $A_j^{(\nu)}$ are admissible operators of commutator type $\geq j$.

Proof. We use a partition of unity and suppose X has arbitrarily small support on a coordinate patch near a boundary point in which we have an orthogonal frame $\omega^1, \ldots, \omega^n$ of $(1,0)$-forms with $\partial r = \gamma\omega^n$, as described above with $L_1, \ldots, L_n$ comprising the dual frame. We have $L_1, \ldots, L_{n-1}$, $\overline{L}_1, \ldots, \overline{L}_{n-1}$, and $Y = L_n - \overline{L}_n$ as tangential vector fields, and $N = L_n + \overline{L}_n$ a normal vector field.

We have the decomposition of the tangential vector field X

$$X = \sum_{j=0}^{n-1} a_j L_j + \sum_{j=0}^{n-1} b_j \overline{L}_j + aY + brN,$$

where the a_j, b_j, a, and b are smooth with compact support. We then prove the theorem for each term in the decomposition.

Case 1). $X = a_jL_j$ or $b_j\overline{L}_j$, $j \leq n-1$, or aY.

We write

$$\gamma^* X^z \mathcal{A}_1 = -\gamma X^\zeta \mathcal{A}_1 + (\gamma X^\zeta + \gamma^* X^z)\mathcal{A}_1.$$

Then an integration by parts gives

$$\gamma^* X^z A_1 f = -A_1(\widetilde{X}^\zeta \gamma f) + (f, (\gamma X^\zeta + \gamma^* X^z)\mathcal{A}_1).$$

We now use the following relations

$$\begin{aligned}(\gamma X^{\zeta}+\gamma^{*}X^{z})\mathcal{E}_{j,\alpha} &= \mathcal{E}_{j,\alpha}\\ (\gamma X^{\zeta}+\gamma^{*}X^{z})\mathcal{E}^{*}_{j,\beta} &= \mathcal{E}^{*}_{j,\beta}\\ (\gamma X^{\zeta}+\gamma^{*}X^{z})P &= \mathcal{E}_{2,0}+\frac{rr^{*}}{\gamma\gamma^{*}}\mathcal{E}_{0,0}\\ &= \mathcal{E}_{0,0}P+\mathcal{E}_{2,0}\\ (\gamma X^{\zeta}+\gamma^{*}X^{z})\phi &= \mathcal{E}_{1,1}+\mathcal{E}_{2,0}.\end{aligned} \tag{3.1}$$

Any type 1 kernel

$$\mathcal{A}_1(\zeta,z)=R_N R_M^{*}\mathcal{E}_{j,\alpha}\mathcal{E}^{*}_{k,\beta}P^{-t_0}\phi^{t_1}\overline{\phi}^{t_2}\phi^{*t_3}\overline{\phi}^{*t_4}r^{l}r^{*m} \tag{3.2}$$

can be decomposed into terms

$$\mathcal{A}_1=\mathcal{A}_1'+\mathcal{A}_2$$

where $\mathcal{A}_1'$ is of *pure* type, meaning it has a representation as in (3.2) but with $t_3=t_4=0$ and $t_1t_2\leq 0$, [4]. From the relations (3.1) we have

$$(\gamma X^{\zeta}+\gamma^{*}X^{z})\mathcal{A}_2=\gamma\mathcal{A}_1+\mathcal{A}_2.$$

In calculating $(\gamma X^{\zeta}+\gamma^{*}X^{z})\mathcal{A}_1'$, we find the term that is not immediately seen to be of type $\mathcal{A}_1$ is that which results from the operator $\gamma X^{\zeta}+\gamma^{*}X^{z}$ falling on ϕ^{t_1}, in which case we obtain the term of double type $(0,0)$

$$B:=R_N R_M^{*}\mathcal{E}_{j+1,\alpha+1}\mathcal{E}^{*}_{k,\beta}P^{-t_0}\phi^{t_1-1}\overline{\phi}^{t_2}r^{l}r^{*m},$$

where $N+\alpha\geq 2$, plus a term which is $\mathcal{A}_1$. We follow [3] to reduce to the case where B can be written as a sum of terms B_σ such that B_σ or $\overline{B}_\sigma$ is of the form

$$\gamma^{2}\phi^{\sigma}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_N R_M^{*}\mathcal{E}_{j+1,\alpha-1}\mathcal{E}^{*}_{k,\beta}P^{-t_0}r^{l}r^{*m},$$

where $\tau_1+\tau_2\leq -3$ and $\tau_1\leq\sigma\leq\tau_1+\tau_2$ or $\tau_2\leq\sigma\leq\tau_1+\tau_2$.

We fix a point z and choose local coordinates ζ such that

$$d\zeta_j(z)=\omega_j(z).$$

Working in a neighborhood of a singularity in the boundary (where we can use a coordinate system as in (3.11) below), we see $\frac{\partial}{\partial\zeta_n}$ is a combination of derivatives with coefficients of the form $\xi_0(z)$, while L_n is a combination of derivatives with coefficients of the form $\xi_0(\zeta)$, where ξ_0 is defined in (2.1). We have $\Lambda_n-\frac{\partial}{\partial z_n}$ is a sum of terms of the form

$$(\xi_0(z)-\xi_0(\zeta))\Lambda^{\epsilon}=\mathcal{E}_{1,-1}\Lambda^{\epsilon},$$

where Λ is a first order differential operator, and the equality follows from

$$\begin{aligned}\frac{1}{\gamma(\zeta)}-\frac{1}{\gamma(z)} &= \frac{\gamma(z)-\gamma(\zeta)}{\gamma(\zeta)\gamma(z)}\\ &= \frac{1}{\gamma(z)}\frac{\gamma^2(z)-\gamma^2(\zeta)}{\gamma(\zeta)(\gamma(\zeta)+\gamma(z))}\\ &= \frac{1}{\gamma(z)}\frac{\xi_1(\zeta)\mathcal{E}_1}{\gamma(\zeta)(\gamma(\zeta)+\gamma(z))}\\ &= \frac{1}{\gamma(z)}\frac{\mathcal{E}_{1,0}}{(\gamma(\zeta)+\gamma(z))}\\ &\lesssim \frac{1}{\gamma(z)}\frac{\mathcal{E}_{1,0}}{\gamma(z)}\\ &= \mathcal{E}_{1,-2}.\end{aligned}$$

Using these special coordinates, we note

$$\begin{aligned}Y\phi &= \gamma+\mathcal{E}_{1,0}+\mathcal{E}_{2,-1}\\ Y\overline{\phi} &= -\gamma+\mathcal{E}_{1,0}+\mathcal{E}_{2,-1}\\ YP &= \mathcal{E}_{1,0}+\frac{\mathcal{E}_{0,0}}{\gamma}\left(P+\mathcal{E}_{2,0}\right)\end{aligned}$$

and write

$$\begin{aligned}B_\sigma =&\gamma^2\phi^\sigma(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+1,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ =&\gamma Y\left(\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+1,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\right)\\ &+\gamma\phi^\sigma(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+2,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^\sigma(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+3,\alpha-2}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma-1}R_NR_M^*\mathcal{E}_{j+2,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma-1}R_NR_M^*\mathcal{E}_{j+3,\alpha-2}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_{N-1}R_M^*\mathcal{E}_{j+1,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\gamma\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+2,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0-1}r^lr^{*m}\\ &+\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+1,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0}r^lr^{*m}\\ &+\phi^{\sigma+1}(\phi+\overline{\phi})^{\tau_1+\tau_2-\sigma}R_NR_M^*\mathcal{E}_{j+3,\alpha-1}\mathcal{E}^*_{k,\beta}P^{-t_0-1}r^lr^{*m}.\end{aligned}$$

Thus

$$B_\sigma = \gamma Y\mathcal{A}_{(1,2)}+\mathcal{A}'_1.$$

By the strict pseudoconvexity of D there exists allowable vector fields W_1, W_2, and W_3, and a function φ, smooth on the interior of D which satisfies

$$\Phi^k\varphi = \mathcal{E}_{0,1-k},$$

where Φ is a first order differential operator, such that Y can be written

$$Y = \varphi[W_1,W_2]+W_3.$$

Thus

$$\begin{aligned}\gamma Y\mathcal{A}_{(2,1)} &= \gamma\varphi[W_1,W_2]\mathcal{A}_{(1,2)} + \gamma W_3\mathcal{A}_{(1,2)}\\ &= \gamma[W_1,W_2]\varphi\mathcal{A}_{(1,2)} + \mathcal{A}'_1\end{aligned}$$

with $\mathcal{A}'_1$ of commutator type ≥ 1.

An integration by parts gives

$$(f,\gamma[W_1,W_2](\varphi\mathcal{A}_2)) = (\widetilde{W}_1\gamma f, W_2(\varphi\mathcal{A}_{(1,2)})) - (\widetilde{W}_2\gamma f, W_1(\varphi\mathcal{A}_{(1,2)})).$$

$\widetilde{W}_1$ and $\widetilde{W}_2$ are allowable vector fields while $W_2(\varphi\mathcal{A}_{(1,2)})$ and $W_1(\varphi\mathcal{A}_{(1,2)})$ are of the form $\mathcal{A}'_1$ where $\mathcal{A}'_1$ is of commutator type, proving the theorem for Case 1.
Case 2). $X = \mathcal{E}_0 rN$.

We use

$$\begin{aligned}\left(r\gamma N^\zeta + r^*\gamma^* N^z\right)\mathcal{E}_{j,\alpha} &= \mathcal{E}_{j,\alpha}\\ \left(r\gamma N^\zeta + r^*\gamma^* N^z\right)P &= \mathcal{E}_{2,0} + \frac{r}{\gamma}\frac{r^*}{\gamma^*}\mathcal{E}_{0,0}\\ &= \mathcal{E}_{2,0} + P\mathcal{E}_{0,0}\\ \left(r\gamma N^\zeta + r^*\gamma^* N^z\right)\phi &= r\mathcal{E}_{0,0} + r^*\mathcal{E}_{0,0}.\end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned}\gamma^* XA_1 f =& (\mathcal{E}_0 r^* f, \gamma^* N^z\mathcal{A}_1)\\ =& (-\mathcal{E}_0 rf, \gamma N^\zeta\mathcal{A}_1) + (f, \mathcal{E}_0\left(r\gamma N^\zeta + r^*\gamma^* N^z\right)\mathcal{A}_1)\\ =& (-\widetilde{N}^\zeta(\mathcal{E}_0 r\gamma f), \mathcal{A}_1) + (f, \mathcal{E}_0\left(r\gamma N^\zeta + r^*\gamma^* N^z\right)\mathcal{A}_1).\end{aligned}$$

We have

$$\widetilde{N}^\zeta(\mathcal{E}_0 r\gamma f) = \mathcal{E}_{0,0} f + \mathcal{E}_0 r\widetilde{N}^\zeta\gamma f$$

and $\mathcal{E}_0 r\widetilde{N}^\zeta$ is an allowable vector field. The relations in (3.3) show that

$$\left(r\gamma N^\zeta + r^*\gamma^* N^z\right)\mathcal{A}_1$$

is of commutator type ≥ 1. Case 2 therefore follows. $\square$

Below we use a criterion for Hölder continuity given by Schmalz (see Lemma 4.1 in [6]) which states

Lemma 3.5 (Schmalz)**.** *Let $D \subseteq \mathbb{R}^m$, $m \geq 1$ be an open set and let $B(D)$ denote the space of bounded functions on D. Suppose r is a C^2 function on $\mathbb{R}^m$, $m \geq 1$, such that $D := \{r < 0\} \subseteq \mathbb{R}^m$. Then there exists a constant $C < \infty$ such that the following holds: If a function $u \in B(D)$ satisfies for some $0 < \alpha \leq 1/2$ and for all $z, w \in D$ the estimate*

$$|u(z) - u(w)| \leq |z-w|^\alpha + \max_{y=z,w}\frac{|\nabla r(y)||z-w|^{1/2+\alpha}}{|r(y)|^{1/2}}$$

then

$$|u(z) - u(w)| \leq C|z-w|^\alpha$$

for all $z, w \in D$.

We will also refer to a lemma of Schmalz (Lemma 3.2 in [6]) which provides a useful coordinate system in which to prove estimates.

Lemma 3.6. *Define x_j by $\zeta_j = x_j + ix_{j+n}$ for $1 \le j \le n$. Let $E_\delta(z) := \{\zeta \in D : |\zeta - z| < \delta\gamma(z)\}$ for $\delta > 0$. Then there is a constant c and numbers l, $m \in \{1, \ldots, 2n\}$ such that for all $z \in D$,*

$$\{-r(\zeta),\ Im\phi(\cdot, z), x_1, \ldots_{\hat{l}}\,,_{\hat{m}} \ldots, x_{2n}\},$$

where x_l and x_m are omitted, forms a coordinate system in $E_c(z)$. We have the estimate

$$dV \lesssim \frac{1}{\gamma(z)^2} \left| dr(\zeta) \wedge d\ Im\phi(\cdot, z) \wedge dx_1 \wedge \ldots_{\hat{l}}\,,_{\hat{m}} \ldots \wedge dx_{2n} \right| \quad \text{on } E_c(z),$$

where dV is the Euclidean volume form on $\mathbb{R}^{2n}$.

We define the function spaces with which we will be working.

Definition 3.7. Let $0 \le \beta$ and $0 \le \delta$. We define

$$\|f\|_{L^{\infty,\beta,\delta}(D)} = \sup_{\zeta \in D} |f(\zeta)| \gamma^\beta(\zeta) |r(\zeta)|^\delta.$$

Definition 3.8. We set for $0 < \alpha < 1$

$$\Lambda_\alpha(D) = \{f \in L^\infty(D) \mid \|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup \frac{|f(\zeta) - f(z)|}{|\zeta - z|^\alpha} < \infty\}.$$

We also define the spaces $\Lambda_{\alpha,\beta}$ by

$$\Lambda_{\alpha,\beta} := \{f : \|f\|_{\Lambda_{\alpha,\beta}} = \|\gamma^\beta f\|_{\Lambda_\alpha} < \infty\}.$$

From [1], we have the

Lemma 3.9.

$$\frac{r_\epsilon}{\gamma} \in C^1(D_\epsilon)$$

with C^1-estimates independent of ϵ.

For our C^k estimates later, we will need the following properties.

Theorem 3.10. *Let T be a smooth first order tangential differential operator on D. For A an operator of type 1 we have*

i) $A : L^{\infty,2+\epsilon,0}(D) \to \Lambda_{\alpha,2-\epsilon'}(D) \qquad 0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1/4$

ii) $\gamma^* TA : L^{\infty,2+\epsilon,0}(D) \to L^{\infty,\epsilon',\delta}(D) \qquad 1/2 < \delta < 1, \quad \epsilon < \epsilon' < 1$

iii) $A : L^{\infty,\epsilon,\delta}(D) \to L^{\infty,\epsilon',0}(D) \qquad \epsilon < \epsilon', \quad \delta < 1/2 + (\epsilon' - \epsilon)/2.$

Proof. i). We will prove *i)* in the cases that $\mathcal{A}$, the kernel of A is of double type $(1,1)$ satisfying the inequality

$$|\mathcal{A}| \lesssim \frac{\gamma(\zeta)^2}{P^{n-1/2-\mu}|\phi|^{\mu+1}} \quad \mu \ge 1$$

and $\mathcal{A}$ is of double type $(1,2)$ satisfying

$$|\mathcal{A}| \lesssim \frac{\gamma(\zeta)}{P^{n-1-\mu}|\phi|^{\mu+1}} \quad \mu \ge 1,$$

all other cases being handled by the same methods.
Case a). $\mathcal{A}$, the kernel of A, is of double type $(1,1)$.

We estimate
(3.4)
$$\int_D \frac{1}{\gamma^\epsilon(\zeta)} \left| \frac{\gamma(z)^{2-\epsilon'}}{(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}} - \frac{\gamma(w)^{2-\epsilon'}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,w)^{n-1/2-\mu}} \right| dV(\zeta).$$
Then the integral in (3.4) is bounded by
$$\begin{aligned}
&\int_D \frac{1}{\gamma^\epsilon(\zeta)} \left| \frac{\gamma(z)^{2-\epsilon'}(\phi(\zeta,w))^{\mu+1} - \gamma(w)^{2-\epsilon'}(\phi(\zeta,z))^{\mu+1}}{(\phi(\zeta,w))^{\mu+1}(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}} \right| dV(\zeta)\\
&\quad + \int_D \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \left| \frac{P(\zeta,z)^{n-1/2-\mu} - P(\zeta,w)^{n-1/2-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}P(\zeta,w)^{n-1/2-\mu}} \right| dV(\zeta)\\
&\quad = I + II.
\end{aligned}$$
In I we use
$$(\phi(\zeta,w))^{\mu+1} - (\phi(\zeta,z))^{\mu+1} = \sum_{l=0}^{\mu} (\phi(\zeta,w))^{\mu-l}(\phi(\zeta,z))^l(\phi(\zeta,w)-\phi(\zeta,z))$$
and
$$\phi(\zeta,w) - \phi(\zeta,z) = O\big(\gamma(\zeta) + |\zeta - z|\big)|z-w|.$$
Therefore
$$\begin{aligned}
I &\lesssim \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{(\gamma(\zeta)+|\zeta-z|)|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-1-2\mu}} dV(\zeta)\\
&\qquad + \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu}} dV(\zeta)\\
&\lesssim \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{3-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-1-2\mu}} dV(\zeta)\\
&\qquad + \sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta)\\
&\qquad + \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu}} dV(\zeta)\\
&= I_a + I_b + I_c
\end{aligned}$$
For the integral I_a we break the region of integration into two parts: $\{|\zeta-w| \le |\zeta-z|\}$ and $\{|\zeta-z| \le |\zeta-w|\}$, and by symmetry we need only consider the region $\{|\zeta-z| \le |\zeta-w|\}$.

We first consider the region E_c, where c is chosen as in Lemma 3.5. Without loss of generality we can choose c sufficiently small so that $\gamma(z) \lesssim \gamma(\zeta)$ holds in $E_c(z)$. We thus estimate
$$\int_{\substack{D\cap E_c\\ |\zeta-z|\le|\zeta-w|}} \gamma(z)^{3-\epsilon'-\epsilon} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-1-2\mu}} dV(\zeta). \tag{3.5}$$
We use $\gamma(z) \lesssim \gamma(w) + |z-w|$ and
$$\begin{aligned}
|z-w|^\beta &\lesssim |\zeta-z|^\beta + |\zeta-w|^\beta\\
&\lesssim |\zeta-w|^\beta
\end{aligned}\tag{3.6}$$

for $\beta > 0$ to bound the integral in (3.5) by a constant times

$$\begin{aligned}
&|z-w|^{1/2+\alpha}\int_{\substack{D\cap E_c\\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^2\gamma(w)|\zeta-w|^{1/2-\alpha}}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-1-2\mu+\epsilon+\epsilon'}}dV(\zeta)\\
&+|z-w|^{\alpha}\int_{\substack{D\cap E_c\\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^2|\zeta-w|^{2-\alpha}}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-1-2\mu+\epsilon+\epsilon'}}dV(\zeta).
\end{aligned}\tag{3.7}$$

We use a coordinate system $s_1, s_2, t_1, \ldots, t_{2n-2}$ as given by Lemma 3.6 with $s_1 = -r(\zeta)$ and $s_2 = \mathrm{Im}\phi$, and the estimate (3.6) on the volume element

$$dV(\zeta) \lesssim \frac{t^{2n-3}}{\gamma(z)^2}|ds_1 ds_2 dt| \tag{3.8}$$

where $t = \sqrt{t_1^2 + \cdots + t_{2n-2}^2}$, and the second line follows from $\gamma(\zeta) \lesssim \gamma(z)$ on $E_c(z)$.

We have the estimates

$$\begin{aligned}
\phi(\zeta,z) &\gtrsim s_1 + |s_2| + t^2\\
\phi(\zeta,w) &\gtrsim -r(w) + s_1 + t^2.
\end{aligned}$$

After redefining s_2 to be positive, we bound the first integral of (3.7) by

$$\begin{aligned}
&\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\times\\
&\qquad\int_V \frac{|\zeta-w|^{1/2-\alpha}}{(s_1+s_2+t^2)^{\mu+1-l}(s_1+|\zeta-w|^2)^{l+1/2}t^{2n-1-2\mu+\epsilon+\epsilon'}}t^{2n-3}ds_1ds_2dt\\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s_1+s_2+t^2)^{\mu+1-l}(s_1+t^2)^{l+1/4+\alpha/2}}ds_1ds_2dt\\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{1}{s_1^{7/8}(s_1+s_2)t^{3/4+\alpha+\epsilon+\epsilon'}}ds_1ds_2dt\\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{1}{s_1^{15/16}s_2^{15/16}t^{3/4+\alpha+\epsilon+\epsilon'}}ds_1ds_2dt\\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w),
\end{aligned}\tag{3.9}$$

where V is a bounded subset of $\mathbb{R}^3$.

The second integral of (3.7) can be bounded by a constant times

$$\begin{aligned}
|z-w|^{\alpha}&\int_V \frac{|\zeta-w|^{2-\alpha}}{(s_1+s_2+t^2)^{\mu+1-l}(s_1+|\zeta-w|^2)^{l+1}t^{2n-1-2\mu+\epsilon+\epsilon'}}t^{2n-3}ds_1ds_2dt\\
&\lesssim |z-w|^{\alpha}\int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s_1+s_2+t^2)^{\mu+1-l}(s_1+t^2)^{l+\alpha/2}}ds_1ds_2dt\\
&\lesssim |z-w|^{\alpha},
\end{aligned}$$

where again V is a bounded subset of $\mathbb{R}^3$. The last line follows by the estimates in (3.9).

In estimating the integrals of I_a over the region $D \setminus E_c$, we write

$$\begin{aligned}
&\int_{\substack{D\setminus E_c\\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-4-2\mu+\epsilon'}} dV(\zeta)\\
&\quad\lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c\\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\zeta-w|^{1-\alpha}}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-4-2\mu+\epsilon'}} dV(\zeta)\\
&\quad\lesssim |z-w|^\alpha \times\\
&\qquad\int_{\substack{D\setminus E_c\\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta-z|^{2n-4-2\mu+\epsilon'}} dV(\zeta)\\
&\quad\lesssim |z-w|^\alpha \int_{D\setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta-z|^{2n-1+\alpha+\epsilon'}} dV(\zeta).
\end{aligned} \tag{3.10}$$

We denote the critical points of r by $p_1, \dots, p_k$, and take ε small enough so that in each

$$U_{2\varepsilon}(p_j) = \{\zeta : D \cap |\zeta - p_j| < 2\varepsilon\},$$

for $j = 1, \dots, k$, there are coordinates $u_{j_1}, \dots, u_{j_m}, v_{j_{m+1}}, \dots, v_{j_{2n}}$ such that

$$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2, \tag{3.11}$$

with $u_{j_\alpha}(p_j) = v_{j_\beta}(p_j) = 0$ for all $1 \le \alpha \le m$ and $m+1 \le \beta \le 2n$, from the Morse Lemma. Let $U_\varepsilon = \bigcup_{j=1}^k U_\varepsilon(p_j)$. We break the problem of estimating (3.10) into subcases depending on whether $z \in U_\varepsilon$.

Suppose $z \in U_\varepsilon(p_j)$. Define $w_1, \dots, w_{2n}$ by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1 \le \alpha \le m \\ v_{j_\alpha} & \text{for } m+1 \le \alpha \le 2n. \end{cases} \tag{3.12}$$

Let $x_1, \dots, x_{2n}$ be defined by $\zeta_\alpha = x_\alpha + i x_{n+\alpha}$. From the Morse Lemma, the Jacobian of the transformation from coordinates $x_1, \dots, x_{2n}$ to $w_1, \dots, w_{2n}$ is bounded from below and above and thus we have

$$|\zeta - z| \simeq |w(\zeta) - w(z)|$$

for $\zeta,\, z \in U_{2\varepsilon}(p_j)$.

From (3.11) we have $\gamma(z) \gtrsim |w(z)|$, and thus

$$\begin{aligned}
|w(\zeta) - w(z)| &\simeq |\zeta - z|\\
&\gtrsim \gamma(z)\\
&\gtrsim |w(z)|\\
&\ge |w(\zeta)| - |w(\zeta) - w(z)|,
\end{aligned}$$

and we obtain

$$\begin{aligned}
|w(\zeta)| &\lesssim |w(\zeta) - w(z)|\\
&\simeq |\zeta - z|.
\end{aligned}$$

Using $|w(\zeta)| \lesssim \gamma(\zeta)$, we estimate, using the coordinates above

$$\begin{aligned}|z-w|^\alpha \int_{U_\varepsilon \setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}} dV(\zeta) \\ &\lesssim |z-w|^\alpha \int_V \frac{u^{m-1} v^{2n-m-1}}{(u+v)^{2n-1+\alpha+\epsilon'+\epsilon}} \\ &\lesssim |z-w|^\alpha,\end{aligned}$$

where we use $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$, $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$, and V is a bounded set.

In integrating over the region $D \setminus U_\varepsilon$ we have

$$\begin{aligned}|z-w|^\alpha \int_{(D\setminus U_\varepsilon)\setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\alpha+\epsilon'}} dV(\zeta) \\ &\lesssim |z-w|^\alpha \int_{(D\setminus U_\varepsilon)\setminus E_c} \frac{1}{\gamma^\epsilon(\zeta)} dV(\zeta) \lesssim |z-w|^\alpha,\end{aligned}$$

which follows by using the coordinates $w_1, \ldots, w_{2n}$ above.

Subcase *b*). Suppose $z \notin U_\varepsilon$. We have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below, since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma^\epsilon(\zeta)} dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \ldots, w_{2n}$ above.

The region in which $|\zeta - w| \leq |\zeta - z|$ is handled in the same manner, and thus we are finished bounding I_a.

We now estimate I_b, and again, we only consider the region $|\zeta - z| \leq |\zeta - w|$. We first estimate the integrals of I_b over the region $E_c(z)$, where c is chosen as in Lemma 3.6, and sufficiently small so that $|\zeta - z| \lesssim \gamma(\zeta)$. As we chose coordinates for the integrals in I_a, we choose a coordinate system in which $s_1 = -r(\zeta)$ and $s_2 = \mathrm{Im}\phi$ and we use the estimate on the volume element given by (3.8). We thus write

(3.13)

$$\begin{aligned}&\int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \gamma(z)^2 \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l} |\phi(\zeta,w)|^{l+1} |\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\ &\quad \lesssim |z-w|^\alpha \times \\ &\qquad \int_{\substack{D\cap E_c \\ |\zeta-z|\leq|\zeta-w|}} \gamma(z)^2 \frac{1}{|\phi(\zeta,z)|^{\mu+1-l} |\phi(\zeta,w)|^{l+1/2+\alpha/2} |\zeta-z|^{2n-2-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\ &\quad \lesssim |z-w|^\alpha \int_V \frac{t^{2n-3}}{(s_1+s_2+t^2)^{\mu+1-l} (s_1+t^2)^{l+1/2+\alpha/2} t^{2n-2-2\mu+\epsilon+\epsilon'}} ds_1 ds_2 dt \\ &\quad \lesssim |z-w|^\alpha \int_0^M \int_0^N \frac{t^{2\mu-1-\epsilon-\epsilon'}}{(s_1+t^2)^{\mu-l} (s_1+t^2)^{l+1/2+\alpha/2}} ds_1 dt \\ &\quad \lesssim |z-w|^\alpha \int_0^M \int_0^N \frac{1}{s_1^{7/8} t^{1/4+\alpha+\epsilon+\epsilon'}} ds dt \\ &\quad \lesssim |z-w|^\alpha,\end{aligned}$$

where we have redefined the coordinate s_2 to be positive, V is a bounded subset of $\mathbb{R}^3$, and $M, N > 0$ are constants.

The integrals of I_b over the region $D \setminus E_c$ are estimated by (3.10) above.

For the integral I_c we use

$$|\gamma(w)^{2-\epsilon'} - \gamma(z)^{2-\epsilon'}| \lesssim |z-w| \left(\gamma(w)^{1-\epsilon'} + \gamma(z)^{1-\epsilon'}\right)$$

and estimate

$$\int_{\substack{D \\ |\zeta - z| \leq |\zeta - w|}} \frac{1}{\gamma^{\epsilon}(\zeta)} \frac{|z-w| \left(\gamma(w)^{1-\epsilon'} + \gamma(z)^{1-\epsilon'}\right)}{|\phi(\zeta,w)|^{\mu+1} |\zeta - z|^{2n-1-2\mu}} dV(\zeta). \tag{3.14}$$

Let us first consider the case $\gamma(w) \leq \gamma(z)$ and integrate (3.14) over the region E_c. We use a coordinate system $s, t_1, \ldots, t_{2n-1}$, with $s = -r$ and the estimate

$$dV(\zeta) \lesssim \frac{t^{2n-2}}{\gamma(z)} ds dt$$

for $t = \sqrt{t_1^2 + \cdots + t_{2n-1}^2}$. We thus bound (3.14) by

(3.15)

$$\begin{aligned}
&\int_{\substack{D \cap E_c \\ |\zeta - z| \leq |\zeta - w|}} \frac{|z-w| \gamma(z)^{1-\epsilon'}}{|\phi(\zeta,w)|^{\mu+1} |\zeta - z|^{2n-1-2\mu+\epsilon}} dV(\zeta) \\
&\qquad \lesssim |z-w|^{\alpha} \int_{\substack{D \cap E_c \\ |\zeta - z| \leq |\zeta - w|}} \frac{\gamma(z)}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2} |\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\
&\qquad \lesssim |z-w|^{\alpha} \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/2+\alpha/2} t^{2n-1-2\mu+\epsilon+\epsilon'}} ds dt \\
&\qquad \lesssim |z-w|^{\alpha} \int_V \frac{1}{s^{3/4} t^{1/2+\epsilon+\epsilon'+\alpha/2}} ds dt \\
&\qquad \lesssim |z-w|^{\alpha},
\end{aligned}$$

where V is here a bounded region of $\mathbb{R}^2$.

Over the complement of E_c, (3.14) is bounded by

$$\begin{aligned}
|z-w|^{\alpha} &\int_{\substack{D \\ |\zeta - z| \leq |\zeta - w|}} \frac{1}{\gamma^{\epsilon}(\zeta)} \frac{1}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2} |\zeta - z|^{2n-2-2\mu+\epsilon'}} dV(\zeta) \\
&\lesssim |z-w|^{\alpha} \int_D \frac{1}{\gamma^{\epsilon}(\zeta)} \frac{1}{|\zeta - z|^{2n-1+\epsilon'+\alpha}} dV(\zeta) \\
&\lesssim |z-w|^{\alpha},
\end{aligned}$$

which follows from the estimates of (3.10) above.

For the case $\gamma(z) \leq \gamma(w)$ we estimate (3.14) over the region E_c using coordinates as above by

$$\begin{aligned}
(3.16)\qquad &\int_{\substack{D\cap E_c\\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)}\frac{|z-w|\gamma(w)^{1-\epsilon'}}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu}}dV(\zeta)\\
&\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}}\gamma(w)\times\\
&\quad\int_{\substack{D\cap E_c\\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)}\frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-1-2\mu+\epsilon'}}dV(\zeta)\\
&\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/4+\alpha/2}t^{2n-2\mu+\epsilon+\epsilon'}}dsdt\\
&\lesssim \frac{|z-w|^{1/2+\alpha/2}}{|r(w)|^{1/2}}\gamma(w),
\end{aligned}$$

where the last line follows as above. While over the complement of E_c we use $\gamma(w) \lesssim |\zeta - w|$ to bound (3.14) by

$$\begin{aligned}
|z-w|^\alpha &\int_{\substack{D\\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)}\frac{1}{|\phi(\zeta,w)|^{\mu+\epsilon'/2+\alpha/2}|\zeta-z|^{2n-1-2\mu}}dV(\zeta)\\
&\lesssim |z-w|^\alpha\int_D \frac{1}{\gamma^\epsilon(\zeta)}\frac{1}{|\zeta-z|^{2n-1+\epsilon'+\alpha}}dV(\zeta)\\
&\lesssim |z-w|^\alpha.
\end{aligned}$$

We are now done with integral I.

For integral II above we again break the integral into regions $|\zeta - z| \leq |\zeta - w|$ and $|\zeta - w| \leq |\zeta - z|$, and we only consider the region $|\zeta - z| \leq |\zeta - w|$, the other case being handled similarly.

We write

$$\begin{aligned}
&\left(P(\zeta,z)^{1/2}\right)^{2n-1-2\mu} - \left(P(\zeta,w)^{1/2}\right)^{2n-1-2\mu} =\\
&\quad\sum_{l=0}^{2n-2\mu-2}\left(P(\zeta,z)^{1/2}\right)^{2n-2-2\mu-l}\left(P(\zeta,w)^{1/2}\right)^{l}\left(P(\zeta,z)^{1/2}-P(\zeta,w)^{1/2}\right),
\end{aligned}$$

and use

$$\begin{aligned}
\left|P(\zeta,z)^{1/2}-P(\zeta,w)^{1/2}\right| &= \frac{|P(\zeta,z)-P(\zeta,w)|}{P(\zeta,z)^{1/2}+P(\zeta,w)^{1/2}}\\
&\lesssim \frac{|\zeta-z|+\frac{|r(\zeta)|}{\gamma(\zeta)}}{|\zeta-z|}|z-w|\\
&\lesssim \frac{|\zeta-w|+\frac{|r(w)|}{\gamma(w)}}{|\zeta-z|}|z-w|,
\end{aligned}$$

which follows from Lemma 3.9.

We thus estimate

$$\int_{\substack{D\\|\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^{\epsilon}(\zeta)}\left|\frac{P(\zeta,z)^{n-1/2-\mu}-P(\zeta,w)^{n-1/2-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1/2-\mu}P(\zeta,w)^{n-1/2-\mu}}\right|dV(\zeta)$$

$$\lesssim \sum_{l=0}^{2n-2\mu-2}\int_{\substack{D\\|\zeta-z|\le|\zeta-w|}} \frac{\frac{\gamma(w)^{2-\epsilon'}}{\gamma^{\epsilon}(\zeta)}|z-w|\left(|\zeta-z|+\frac{|r(w)|}{\gamma(w)}\right)dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1}\left(P(\zeta,z)^{1/2}\right)^{l+1}\left(P(\zeta,w)^{1/2}\right)^{2n-1-2\mu-l}|\zeta-z|}$$

$$\lesssim \int_{\substack{D\\|\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^{\epsilon}(\zeta)}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}dV(\zeta)$$

$$+\int_{\substack{D\\|\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^{\epsilon}(\zeta)}\frac{|r(w)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n+1-2\mu}}dV(\zeta)$$

$$=II_a+II_b.$$

For II_a, we break the integral into the regions $E_c(z)$ and its complement. We first consider

(3.17)

$$\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^{\epsilon}(\zeta)}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}dV(\zeta)$$

$$\lesssim |z-w|^{\alpha}\int_{\substack{D\setminus E_c\\|\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma^{\epsilon}(\zeta)}\frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2+\epsilon'/2}|\zeta-z|^{2n-2\mu}}dV(\zeta)$$

$$\lesssim |z-w|^{\alpha}\int_D \frac{1}{\gamma^{\epsilon}(\zeta)}\frac{1}{|\zeta-z|^{2n-1+\alpha+\epsilon'}}dV(\zeta)$$

$$\lesssim |z-w|^{\alpha},$$

where we use $\gamma(w)\lesssim|\zeta-w|$ and the estimates for (3.10).

We then bound the integral II_a over the region $E_c(z)$ by considering the different cases $\gamma(w)\le\gamma(z)$ and $\gamma(z)\le\gamma(w)$. In the case $\gamma(w)\le\gamma(z)$, we use a coordinate system, $s,t_1,\ldots,t_{2n-1}$, in which $s=-r(\zeta)$, and using the estimate

$$dV(\zeta)\lesssim\frac{t^{2n-2}}{\gamma(z)}dsdt, \tag{3.18}$$

we have

(3.19)

$$\int_{\substack{D\cap E_c\\|\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^{\epsilon}(\zeta)}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}dV(\zeta)$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_{\substack{D\cap E_c\\|\zeta-z|\le|\zeta-w|}}\gamma(z)^{1-\epsilon'}\frac{dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon}}$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V\frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s+t^2)^{\mu+1/4+\alpha/2}}dsdt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V\frac{1}{s^{7/8}t^{3/4+\alpha+\epsilon+\epsilon'}}dsdt$$

$$\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w).$$

In the case $\gamma(z) \le \gamma(w)$, we estimate as above

$$\begin{aligned}
&\int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}} dV(\zeta) \\
&\quad\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \gamma(w) \frac{dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}} \\
&\quad\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{(\gamma(z)+|\zeta-w|)dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}}.
\end{aligned}$$

The integral involving $\gamma(z)$ is estimated exactly as above. We thus have to deal with

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{|\zeta-w|dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}},$$

which we estimate using the coordinates $s, t_1, \ldots, t_{2n-1}$ above by

$$\begin{aligned}
&\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{|\zeta-w|dV(\zeta)}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta-z|^{2n-2\mu+\epsilon+\epsilon'}} \\
&\qquad\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_V \frac{t^{2n-2}dsdt}{(s+t^2)^{\mu-1/4+\alpha/2}(s+t)^{2n-2\mu+1+\epsilon+\epsilon'}} \\
&\qquad\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w) \int_V \frac{1}{s^{3/4+\alpha/2+\epsilon+\epsilon'+\delta}t^{1-\delta}} dsdt \\
&\qquad\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w),
\end{aligned}$$

where $0 < \delta < 1/4 - (\alpha/2 + \epsilon + \epsilon')$.

For II_b we first estimate

$$\begin{aligned}
&\int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n+1-2\mu}} dV(\zeta) \\
&\qquad\lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma^\epsilon(\zeta)} \frac{|\zeta-w|^{2-\alpha-\epsilon'}}{|\phi(\zeta,w)|^{\mu}|\zeta-z|^{2n+1-2\mu}} dV(\zeta) \\
&\qquad\lesssim |z-w|^\alpha \int_D \frac{1}{\gamma^\epsilon(\zeta)} \frac{1}{|\zeta-z|^{2n-1+\alpha+\epsilon'}} dV(\zeta) \\
&\qquad\lesssim |z-w|^\alpha,
\end{aligned}$$

where c is chosen as in Lemma 3.6 and we use $\gamma(w) \lesssim |\zeta-w|$ on $D \setminus E_c(z)$.

We now finish the estimates for II_b. We have

$$\begin{aligned}
&\int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma^\epsilon(\zeta)} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n+1-2\mu}} dV(\zeta) \\
&\quad\lesssim |z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \gamma(w)^{1-\epsilon'} \frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon}} dV(\zeta).
\end{aligned} \tag{3.20}$$

We again consider the different cases $\gamma(w) \le \gamma(z)$ and $\gamma(z) \le \gamma(w)$ separately. With $\gamma(w) \le \gamma(z)$, we use coordinates $s, t_1, \ldots, t_{2n-1}$ as above with the volume

estimate (3.18) to estimate (3.20) by

$$\begin{aligned}
|z-w|^\alpha \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu-1/2+\alpha/2}(s+t)^{2n+1-2\mu+\epsilon+\epsilon'}} dsdt \\
\lesssim |z-w|^\alpha \int_V \frac{1}{s^{1/2+\alpha/2+\epsilon+\epsilon'+\delta} t^{1-\delta}} dsdt \\
\lesssim |z-w|^\alpha,
\end{aligned}$$

where $0<\delta<1/2-(\alpha/2+\epsilon+\epsilon')$, and V again denotes a bounded subset of $\mathbb{R}^2$.

In the case $\gamma(z)\le\gamma(w)$, we write $\gamma(w)\lesssim\gamma(z)+|\zeta-w|$, and estimate (3.20) by

$$|z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)+|\zeta-w|}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta).$$

The integral involving $\gamma(z)$ is handled exactly as above, so we estimate

$$\begin{aligned}
|z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{|\zeta-w|}{|\phi(\zeta,w)|^{\mu-1/2+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\
\lesssim |z-w|^\alpha \int_{\substack{D\cap E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{|\phi(\zeta,w)|^{\mu-1+\alpha/2}|\zeta-z|^{2n+1-2\mu+\epsilon+\epsilon'}} dV(\zeta).
\end{aligned}$$

The case of $\mu=1$ is trivial so we assume $\mu\ge 2$ and using the coordinates $s,t_1,\ldots,t_{2n-1}$, we estimate

$$\begin{aligned}
|z-w|^\alpha \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu-1+\alpha/2}(s+t)^{2n+2-2\mu+\epsilon+\epsilon'}} dsdt \\
\lesssim |z-w|^\alpha \int_V \frac{1}{s^{3/4+\alpha/2+\epsilon+\epsilon'} t^{1/2}} dsdt \\
\lesssim |z-w|^\alpha.
\end{aligned}$$

Case b). $\mathcal{A}$ is of double type $(1,2)$.

Following the arguments above we see we need to estimate

$$\begin{aligned}
&\int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} \left| \frac{\gamma(z)^{2-\epsilon'}(\phi(\zeta,w))^{\mu+1}-\gamma(w)^{2-\epsilon'}(\phi(\zeta,z))^{\mu+1}}{(\phi(\zeta,w))^{\mu+1}(\phi(\zeta,z))^{\mu+1}P(\zeta,z)^{n-1-\mu}} \right| dV(\zeta) \\
&+\int_D \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \left| \frac{P(\zeta,z)^{n-1-\mu}-P(\zeta,w)^{n-1-\mu}}{(\phi(\zeta,w))^{\mu+1}P(\zeta,z)^{n-1-\mu}P(\zeta,w)^{n-1-\mu}} \right| dV(\zeta) \\
&= III+IV.
\end{aligned}$$

Following the calculations for integral I in case a) we estimate integral III by the integrals

$$\begin{aligned}
&\sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{\epsilon}} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta) \\
&\qquad +\sum_{l=0}^{\mu} \int_D \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta-z|^{2n-3-2\mu}} dV(\zeta) \\
&\qquad +\int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|\gamma(z)^{2-\epsilon'}-\gamma(w)^{2-\epsilon'}|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta) \\
&\qquad = III_a+III_b+III_c.
\end{aligned}$$

Estimates for the integral III_a are given by I_b in case a).

For the integrals of III_b, we consider separately the regions $E_c(z)$ and its complement. We also only consider the case $|\zeta - z| \le |\zeta - w|$.

In the region $D \cap E_c(z)$, we use a coordinate system in which $s = -r(\zeta)$ is a coordinate, and we use the estimate on the volume element in $E_c(z)$ given by (3.18). We can also assume that c is sufficiently small to guarantee that $|\zeta - z| \lesssim \gamma(\zeta)$ in E_c.

The integrals

$$\int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-3-2\mu}} dV(\zeta)$$

can thus be bounded by

$$\begin{aligned}
&\frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z)\times \\
&\int_V \frac{|\zeta - w|^{1/2-\alpha}}{(s+|\zeta - z|^2)^{\mu+1/2-l}(s+|\zeta-w|^2)^{l+1}|\zeta - z|^{2n-2-2\mu+\epsilon+\epsilon'}} t^{2n-2} dsdt \\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z) \int_V \frac{t^{2n-2}}{(s+|\zeta - z|^2)^{\mu+5/4+\alpha/2}|\zeta - z|^{2n-2-2\mu+\epsilon+\epsilon'}} dsdt \\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z) \int_V \frac{t^{2\mu-\epsilon-\epsilon'}}{(s+t^2)^{\mu+5/4+\alpha/2}} dsdt \\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z) \int_V \frac{1}{s^{7/8}t^{3/4+\alpha+\epsilon+\epsilon'}} dsdt \\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(z)|^{1/2}}\gamma(z),
\end{aligned}$$

where V is a bounded subset of $\mathbb{R}^2$.

We now estimate

$$\begin{aligned}
&\int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(z)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,z)|^{\mu+1-l}|\phi(\zeta,w)|^{l+1}|\zeta - z|^{2n-3-2\mu}} dV(\zeta) \\
&\qquad \lesssim |z-w|^{\alpha} \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta - z|^{2n-3-2l+\epsilon'}} dV(\zeta).
\end{aligned}$$

We use coordinates $u_{j_1},\ldots,u_{j_m},v_{j_{m+1}},\ldots,v_{j_{2n}}$ as in (3.11) and the neighborhoods $U_{2\varepsilon}(p_j)$ defined above. We break the problem into subcases depending on whether $z \in U_\varepsilon$.

Subcase a). Suppose $z \in U_\varepsilon(p_j)$. As we did above above define $w_1,\ldots,w_{2n}$ by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1 \le \alpha \le m \\ v_{j_\alpha} & \text{for } m+1 \le \alpha \le 2n, \end{cases}$$

and let $x_1, \dots, x_{2n}$ be defined by $\zeta_\alpha = x_\alpha + ix_{n+\alpha}$. Recall that we have $|w(\zeta)| \lesssim |\zeta - z|$ and $|w(\zeta)| \lesssim \gamma(\zeta)$. Thus we estimate, using the coordinates above,

(3.21)
$$\begin{aligned}
&|z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{l+1/2+\alpha/2}|\zeta-z|^{2n-3-2l+\epsilon'}} dV(\zeta)\\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\zeta-z|^{2n-2+\alpha+\epsilon'}} dV(\zeta)\\
&\qquad \lesssim |z-w|^\alpha \int_V \frac{u^{m-1}v^{2n-m-1}}{(u+v)^{2n-1+\alpha+\epsilon+\epsilon'}} du dv\\
&\qquad \lesssim |z-w|^\alpha \int_V \frac{1}{u^{1/2}v^{1/2+\alpha+\epsilon+\epsilon'}} du dv\\
&\qquad \lesssim |z-w|^\alpha,
\end{aligned}$$

where we use $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$, $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$, and V is a bounded set.

Subcase b). Suppose $z \notin U_\varepsilon$. We have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below, since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma(\zeta)^{1+\epsilon}} dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \dots, w_{2n}$ above.

We now estimate integral III_c. We use

$$|\gamma(z)^{2-\epsilon'} - \gamma(w)^{2-\epsilon'}| \lesssim |z-w| \left(\gamma(z)^{1-\epsilon'} + \gamma(w)^{1-\epsilon'}\right)$$

to write

$$III_c \lesssim \int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'} + \gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta).$$

We first assume $\gamma(w) \le \gamma(z)$. Then we estimate

$$\int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta). \tag{3.22}$$

by breaking the integral into the regions E_c and $D \setminus E_c$. In E_c, again assuming c is sufficiently small so that $|\zeta - z| \lesssim \gamma(\zeta)$, (3.22) is bounded by

$$\int_{\substack{D \\ |\zeta-z|\le|\zeta-w|}} \gamma(z)^{1-\epsilon'} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-1-2\mu+\epsilon}} dV(\zeta),$$

which we showed to be bounded by $|z-w|^\alpha$ in (3.15). In the region $D \setminus E_c$, we estimate

$$\begin{aligned}
&\int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{\gamma(z)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2-2\mu}} dV(\zeta)\\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta-z|\le|\zeta-w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{\mu+1/2+\alpha/2}|\zeta-z|^{2n-3-2\mu+\epsilon'}} dV(\zeta)\\
&\qquad \lesssim |z-w|^\alpha,
\end{aligned}$$

where the last line follows from (3.21) above.

We therefore now consider the case $\gamma(z) \le \gamma(w)$ so that

$$III_c \lesssim \int_{\substack{D \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} dV(\zeta).$$

In the region E_c we estimate

$$\begin{aligned}
&\int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} dV(\zeta) \\
&\quad \lesssim \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(w) \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\
&\quad \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-1-2\mu+\epsilon+\epsilon'}} dV(\zeta).
\end{aligned}$$

Using the coordinate system $s = -r(\zeta),\ t_1 \ldots, t_{2n-2}$ with volume estimate (3.18) as above we can estimate

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2n-2}}{(s+t^2)^{\mu+1/4+\alpha/2} t^{2n-2\mu+\epsilon+\epsilon'}} dsdt \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

by (3.19).

In the region $D \setminus E_c$, we use $\gamma(w) \lesssim |\zeta - w|$ to estimate

(3.23)

$$\begin{aligned}
&\int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{1-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} dV(\zeta) \\
&\qquad \lesssim \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|\zeta - w|^{1-\epsilon'}|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2-2\mu}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{\mu+\epsilon'/2+\alpha/2}|\zeta - z|^{2n-2-2\mu}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\zeta - z|^{2n-2+\epsilon'+\alpha}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha \int_V \frac{u^{m-1} v^{2n-1-m}}{(u+v)^{2n-1+\epsilon+\epsilon'+\alpha}} dudv \\
&\qquad \lesssim |z-w|^\alpha,
\end{aligned}$$

where the coordinates u and v are defined as in (3.21), and where the last line follows from (3.21). We are now done estimating integral III and we turn to IV.

As in case $a)$ for integral II we estimate IV by the integrals

$$\begin{aligned}
&\int_{\substack{D \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta) \\
&\quad + \int_{\substack{D \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{2+\epsilon}} \frac{|r(\zeta)||z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-2\mu}} dV(\zeta) \\
&= IV_a + IV_b.
\end{aligned}$$

To estimate IV_a we break the region of integration in E_c and $D \setminus E_c$. In the region $D \setminus E_c$ we use $\gamma(w) \lesssim |\zeta - w|$ and estimate

$$\begin{aligned}
&\int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+\epsilon'/2}|\zeta - z|^{2n-1-2\mu}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\phi(\zeta,w)|^{\mu+\epsilon'/2-1/2+\alpha/2}|\zeta - z|^{2n-1-2\mu}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha \int_{\substack{D\setminus E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{\gamma(\zeta)^{1+\epsilon}} \frac{1}{|\zeta - z|^{2n-2-2\mu+\epsilon'+\alpha}} dV(\zeta) \\
&\qquad \lesssim |z-w|^\alpha,
\end{aligned}$$

where the last line follows from (3.23).

In the region E_c we consider the different cases $\gamma(w) \le \gamma(z)$ and $\gamma(z) \le \gamma(w)$ separately. In the case $\gamma(w) \le \gamma(z)$, we write

$$\begin{aligned}
&\int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta) \\
&\qquad \lesssim \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(z)}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu+\epsilon'}} dV(\zeta) \\
&\qquad \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(z) \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} dV(\zeta)
\end{aligned}$$

and we choose a coordinate system in which $s = -r(\zeta)$ and we use the estimate on the volume element given by (3.18) to reduce the estimate to

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_V \frac{t^{2\mu-2-\epsilon-\epsilon'}}{(s+t^2)^{\mu+1/4+\alpha/2}} ds dt \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)$$

which follows from (3.19).

In the case $\gamma(z) \le \gamma(w)$ we have

$$\begin{aligned}
&\int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{1+\epsilon}} \frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta - z|^{2n-1-2\mu}} dV(\zeta) \\
&\lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(w) \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} dV(\zeta).
\end{aligned}$$

We then write $\gamma(w) \lesssim \gamma(z) + |\zeta - w|$, and we bound

$$\begin{aligned}
&\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \gamma(z) \frac{1}{|\phi(\zeta,w)|^{\mu+1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} dV(\zeta) \\
&\qquad\qquad \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w)
\end{aligned}$$

by (3.16) and then consider

$$\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}} \gamma(w) \int_{\substack{D\cap E_c \\ |\zeta - z| \le |\zeta - w|}} \frac{1}{|\phi(\zeta,w)|^{\mu-1/4+\alpha/2}|\zeta - z|^{2n-2\mu+\epsilon+\epsilon'}} dV(\zeta). \tag{3.24}$$

The case $\mu = 1$ is trivial so we assume $\mu \geq 2$ in which case we use coordinates $s = -r(\zeta), t_1, \ldots, t_{2n-1}$ and bound (3.24) by

$$\begin{aligned}\frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{t^{2\mu-3-\epsilon-\epsilon'}}{(s+t^2)^{\mu-1/4+\alpha/2}}dsdt &\\ \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w)\int_V \frac{1}{s^{7/8}t^{3/4+\alpha+\epsilon+\epsilon'}}dsdt &\\ \lesssim \frac{|z-w|^{1/2+\alpha}}{|r(w)|^{1/2}}\gamma(w). &\end{aligned}$$

To estimate IV_b we use

$$\frac{|r(\zeta)|}{\gamma(\zeta)^2} \lesssim 1,$$

which follows by working in the coordinates of (3.11) near a critical point, and thus we have

$$IV_b \lesssim \int_{\substack{D\\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{\epsilon}}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}dV(\zeta). \tag{3.25}$$

We break the regions of integration in (3.25) into E_c and $D\setminus E_c$. The estimates for IV_b in the region E_c are handled in the manner as was done for IV_a. In the region $D\setminus E_c$ we use $\gamma(w)\lesssim|\zeta-w|$ to bound (3.25) by

$$\begin{aligned}&\int_{\substack{D\setminus E_c\\ |\zeta-z|\leq|\zeta-w|}} \frac{\gamma(w)^{2-\epsilon'}}{\gamma(\zeta)^{\epsilon}}\frac{|z-w|}{|\phi(\zeta,w)|^{\mu+1}|\zeta-z|^{2n-2\mu}}dV(\zeta)\\
&\qquad\lesssim |z-w|^{\alpha}\int_{\substack{D\setminus E_c\\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma(\zeta)^{\epsilon}}\frac{1}{|\phi(\zeta,w)|^{\mu-1/2+\epsilon'/2+\alpha/2}|\zeta-z|^{2n-2\mu}}dV(\zeta)\\
&\qquad\lesssim |z-w|^{\alpha}\int_{\substack{D\setminus E_c\\ |\zeta-z|\leq|\zeta-w|}} \frac{1}{\gamma(\zeta)^{\epsilon}}\frac{1}{|\zeta-z|^{2n-1+\epsilon'+\alpha}}dV(\zeta)\\
&\qquad\lesssim |z-w|^{\alpha}.\end{aligned}$$

ii). For T^z a smooth first order tangential differential operator on D, with respect to the z variable, we have

$$\begin{aligned}T^z r &= 0\\
T^z r^* &= \mathcal{E}_{0,0} r\\
T^z P &= \mathcal{E}_{1,0} + \mathcal{E}_{0,0}\frac{r}{\gamma}\frac{r^*}{(\gamma^*)^2}\\
&= \mathcal{E}_{1,0} + \frac{\mathcal{E}_{0,0}}{\gamma^*}(P+\mathcal{E}_{2,0})\\
T^z\phi &= \mathcal{E}_{0,1}+\mathcal{E}_{1,0}.\end{aligned}$$

We consider first the case in which the kernel of A is of double type $(1,3)$, of the form $\mathcal{A}_{(3)}(\zeta,z)$, where the subscript (3) refers to the smooth type.

Thus we write

$$\gamma^* T^z \mathcal{A}_{(3)} = \gamma^*\mathcal{A}_{(1)}\gamma + \gamma^*\mathcal{A}_{(2)} + \mathcal{A}_{(3)}, \tag{3.26}$$

and estimate integrals involving the various forms the integral kernels of different types assume.

We insert (3.26) into

$$\gamma^* T A_{(3)} f = \int_D f(\zeta)\gamma^* T^z \mathcal{A}_{(3)}(\zeta, z) dV(\zeta)$$

and we change the factors of γ^* through the equality $\gamma(z) = \gamma(\zeta) + \mathcal{E}_{1,0}$. $ii)$ will then follow in this case by the estimates

$$\begin{aligned}
&\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^{\epsilon}(\zeta)} |\mathcal{A}_{(1)}(\zeta, z)| dV(\zeta) \lesssim \frac{1}{|r(z)|^{\delta}} \\
&\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^{1+\epsilon}(\zeta)} |\mathcal{A}_{(2)}(\zeta, z)| dV(\zeta) \lesssim \frac{1}{|r(z)|^{\delta}} \\
&\int_D \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} |\mathcal{A}_{(3)}(\zeta, z)| dV(\zeta) \lesssim \frac{1}{|r(z)|^{\delta}}.
\end{aligned} \tag{3.27}$$

We will prove the case of (3.27) in which $\mathcal{A}_{(3)}$ satisfies

$$|\mathcal{A}_{(3)}| \lesssim \frac{1}{P^{n-3/2-\mu}|\phi|^{\mu+1}} \qquad \mu \geq 1.$$

The other cases are handled similarly.

Using the notation from $i)$ above, we choose coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ such that

$$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2,$$

and let $U_\varepsilon = \bigcup_{j=1}^k U_\varepsilon(p_j)$. We break the problem into subcases depending on whether $z \in U_\varepsilon$.

Subcase a). Suppose $z \in U_\varepsilon(p_j)$. We estimate

$$\int_{U_{2\varepsilon}(p_j)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} dV(\zeta) \tag{3.28}$$

and

$$\int_{D_\epsilon \setminus U_{2\varepsilon}} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1} P^{n-3/2-\mu}} dV(\zeta). \tag{3.29}$$

We break up the integral in (3.28) into integrals over $E_c(z)$ and its complement, where c is as in Lemma 3.6. We also choose $c < 1$ so that we also have the estimate $|\zeta - z| \lesssim \gamma(\zeta)$.

We set $\theta = -r(z)$.

In the case $U_{2\varepsilon}(p_j)\cap E_c(z)$, we use a coordinate system, $s = -r(\zeta)$, $t_1, \dots, t_{2n-1}$, and estimate

$$\begin{aligned}
&\int_{U_{2\varepsilon}(p_j)\cap E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1}P^{n-3/2-\mu}} dV(\zeta) \\
&\qquad\lesssim \int_V \frac{t^{2n-2}}{\gamma^{1-\epsilon'}(z)(\theta+s+t^2)^{\mu+1}(s+t)^{2n-1-2\mu+\epsilon}} dsdt \\
&\qquad\lesssim \int_V \frac{t^{2\mu-2+\epsilon'-\epsilon}}{(\theta+s+t^2)^{\mu+1}} dsdt \\
&\qquad\lesssim \frac{1}{\theta^\delta} \int_V \frac{t^{2\mu-2+\epsilon'-\epsilon}}{(s+t^2)^{\mu+1-\delta}} dsdt \\
&\qquad\lesssim \frac{1}{\theta^\delta} \int_0^M \frac{1}{s^{3/2-\delta}} ds \int_0^\infty \frac{\tilde{t}^{2\mu-2+\epsilon'-\epsilon}}{(1+\tilde{t}^2)^{\mu+1-\delta}} d\tilde{t} \\
&\qquad\lesssim \frac{1}{\theta^\delta},
\end{aligned}$$

where $M > 0$ is some constant, and we make the substitution $t = s^{1/2}\tilde{t}$.

We now estimate the integral

$$\int_{U_{2\varepsilon}(p_j)\setminus E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1}P^{n-3/2-\mu}} dV(\zeta). \tag{3.30}$$

Defining $u = \sqrt{u_{j_1}^2 + \cdots + u_{j_m}^2}$, $v = \sqrt{v_{j_{m+1}}^2 + \cdots + v_{j_{2n}}^2}$, and using the estimates from above

$$\begin{aligned}
|w(\zeta)| &\lesssim |\zeta - z| \\
|w(\zeta)| &\lesssim \gamma(\zeta),
\end{aligned}$$

where $w(\zeta)$ is defined as in (3.12), we can bound the integral in (3.30) by

$$\begin{aligned}
&\int_{U_{2\varepsilon}(p_j)\setminus E_c(z)} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1}P^{n-3/2-\mu}} dV(\zeta) \\
&\qquad\lesssim \int_V \frac{u^{m-1}v^{2n-m-1}}{(u+v)^{2n-1+\epsilon-\epsilon'}(\theta+u^2+v^2)} dudv \\
&\qquad\lesssim \int_V \frac{1}{(u+v)^{1+\epsilon-\epsilon'}(\theta+u^2+v^2)} dudv \\
&\qquad\lesssim \frac{1}{\theta^\delta} \int_V \frac{1}{(u+v)^{3-2\delta+\epsilon-\epsilon'}} dudv \\
&\qquad\lesssim \frac{1}{\theta^\delta},
\end{aligned} \tag{3.31}$$

where V is a bounded region. We have therefore bounded (3.28), and we turn now to (3.29).

In $D\setminus U_{2\varepsilon}$ we have that $|\zeta - z|$ and $\gamma(\zeta)$ are bounded from below so

$$\int_{D\setminus U_{2\varepsilon}} \frac{\gamma^{\epsilon'}(z)}{\gamma^{2+\epsilon}(\zeta)} \frac{1}{|\phi|^{\mu+1}P^{n-3/2-\mu}} dV(\zeta) \lesssim 1.$$

This finishes subcase a).

Case b). Suppose $z \notin U_\varepsilon$. We divide D into the regions $D\cap E_c(z)$ and $D\setminus E_c(z)$.

In $D \cap E_c(z)$ the same coordinates and estimates work here as in establishing the estimates for the integral in (3.31).

In $D \setminus E_c(z)$ we have $|\zeta - z| \gtrsim \gamma(z)$, but $\gamma(z)$ is bounded from below, since $z \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma^{2+\epsilon}(\zeta)} dV(\zeta),$$

which is easily done by working with the coordinates $w_1, \ldots, w_{2n}$ above.

iii). The proof of *iii*) follows the same steps as those in the proof of *ii*), and we leave the details to the reader. □

Theorem 3.11. *Let X be a smooth tangential vector field. Then*

$$\gamma^* X^z E_{1-2n} = -E_{1-2n} \tilde{X}^\zeta \gamma + E^{(0)}_{1-2n} + \sum_{\nu=1}^{l} E^{(\nu)}_{1-2n},$$

where $\tilde{X}$ is the adjoint of X and the $E^{(\nu)}_{1-2n}$ are isotropic operators.

Proof. The proof follows the line of argument used in proving case 1) of Theorem 3.4, and makes use of $(\gamma X^\zeta + \gamma^* X^z)\mathcal{E}^i_{1-2n} = \mathcal{E}^i_{1-2n}$. □

Theorem 3.12. *Let T be a smooth tangential vector field. Set E to be an operator with kernel of the form $\mathcal{E}^i_{1-2n}(\zeta, z) R_1(\zeta)$ or $\mathcal{E}^i_{2-2n}(\zeta, z)$. Then we have the following properties:*

i) $E_{1-2n} : L^p(D) \to L^s(D)$

ii) $E : L^{\infty, 2+\epsilon, 0}(D) \to \Lambda_{\alpha, 2-\epsilon'}(D) \qquad 0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1$

iii) $\gamma^* T E : \Lambda_{\alpha, 2+\epsilon}(D) \to L^{\infty, \epsilon', 0}(D) \qquad \epsilon < \epsilon'$

iv) $E : L^{\infty, \epsilon, \delta}(D) \to L^{\infty, \epsilon', 0}(D) \qquad \epsilon < \epsilon', \quad \delta < 1/2 + (\epsilon' - \epsilon)/2$

for any $1 \le p \le s \le \infty$ with $1/s > 1/p - 1/2n$.

Proof. *i*) is presented in [3].

The proof of *ii*) follows that of Theorem 3.10 *i*).

For *iii*) we let $\mathcal{E}(\zeta, z)$ be the kernel of E, and we calculate

$$\begin{aligned}
(\gamma^*)^{1+\epsilon'} T E f &= \int_D f(\zeta) \gamma^* T^z \mathcal{E}(\zeta, z) dV(\zeta) \\
&= \int_D (\gamma^*)^{\epsilon'} \gamma^{2+\epsilon} f(\zeta) \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} dV(\zeta) \\
&= \int_D (\gamma^*)^{\epsilon'} (\gamma^{2+\epsilon} f(\zeta) - (\gamma^*)^{2+\epsilon} f(z)) \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} dV(\zeta) \\
&\quad + (\gamma^*)^{2+\epsilon} f(z) \int_D (\gamma^*)^{\epsilon'} \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} dV(\zeta).
\end{aligned} \tag{3.32}$$

We use Theorem 3.11 in the last integral to bound the last term of (3.32) by

$$\begin{aligned}
(\gamma^*)^{2+\epsilon} f(z) \int_D \mathcal{E}_{1-2n,0} (\gamma^*)^{\epsilon'} \left(\frac{1}{\gamma^{1+\epsilon}} + \frac{\mathcal{E}_{1,0}}{\gamma^{2+\epsilon}} \right) dV(\zeta) &\lesssim (\gamma^*)^{2+\epsilon} |f(z)| \\
&\lesssim \|f\|_{L^{\infty, 2+\epsilon, 0}}, \\
&\lesssim \|f\|_{\Lambda_{\alpha, 2+\epsilon}},
\end{aligned}$$

where the first inequality can be proved by breaking the integrals into the regions $U_{2\varepsilon}$ and $D \setminus U_{2\varepsilon}$ and in the region $D \setminus U_{2\varepsilon}$ using the same coordinates as in the proof of Theorem 3.10 *ii*).

For the first integral in (3.32), we note if $f \in \Lambda_\alpha$ then $\gamma^{2+\epsilon} f \in \Lambda_\alpha$. We have

$$\int_D (\gamma^*)^{\epsilon'} \left| \left(\gamma^{2+\epsilon} f(\zeta) - (\gamma^*)^{2+\epsilon} f(z)\right) \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} \right| dV(\zeta)$$
$$\lesssim \|\gamma^{2+\epsilon} f\|_{\Lambda_\alpha} \int_D |\zeta - z|^\alpha (\gamma^*)^{\epsilon'} \left| \frac{\gamma^* T^z \mathcal{E}(\zeta, z)}{\gamma^{2+\epsilon}} \right| dV(\zeta)$$
$$\lesssim \|\gamma^{2+\epsilon} f\|_{\Lambda_\alpha}.$$

The proof of *iv*) follow as in the case of Theorem 3.10 *iii*). □

4. C^k ESTIMATES

We define Z_1 operators to be those which take the form

$$Z_1 = A_{(1,1)} + E_{1-2n} \circ \gamma,$$

and we write Theorem 2.3 as

$$\gamma^3 f = Z_1 \gamma^2 \bar{\partial} f + Z_1 \gamma^2 \bar{\partial}^* f + Z_1 f. \tag{4.1}$$

We define Z_j operators to be those operators of the form

$$Z_j = \overbrace{Z_1 \circ \cdots \circ Z_1}^{j \text{ times}}.$$

We establish mapping properties for Z_j operators

Lemma 4.1. *For $1 < p < \infty$ and $j \geq 1$*

$$\|Z_j f\|_p \lesssim \|f\|_{L^{p,jp}}, \tag{4.2}$$

and for $0 < \epsilon' < \epsilon$

$$\|Z_j f\|_{L^{\infty,\epsilon,0}} \lesssim \|f\|_{L^{\infty,j+\epsilon',0}}. \tag{4.3}$$

Proof. We prove (4.2) for kernels of the form $\mathcal{A}_{(1,1)}(\zeta, z)$, where $\mathcal{A}_{(1,1)}$ is a kernel of double type $(1,1)$. We show below that $\mathcal{A}_{(1,1)}(\zeta, z)$ satisfies

$$\sup_{z \in \Omega} \int \frac{1}{\gamma(\zeta)} |\mathcal{A}_{(1,1)}(\zeta, z)| |r(\zeta)|^{-\delta} |r(z)|^\delta dV(\zeta) < \infty \tag{4.4}$$

for $\delta < 1$. The lemma then follows from the generalized Young's inequality.

We further restrict our proof to the cases in which $\mathcal{A}_{(1,1)}$ satisfies

$$i)\ \frac{1}{\gamma(\zeta)} |\mathcal{A}_{(1,1)}| \lesssim \gamma(\zeta) \frac{1}{P^{n-1/2-\mu} |\phi|^{\mu+1}} \qquad \mu \geq 1$$
$$ii)\ \frac{1}{\gamma(\zeta)} |\mathcal{A}_{(1,1)}| \lesssim \frac{1}{P^{n-1-\mu} |\phi|^{\mu+1}} \qquad \mu \geq 1$$
$$iii)\ \frac{1}{\gamma(\zeta)} |\mathcal{A}_{(1,1)}| \lesssim \frac{1}{\gamma(\zeta)} \frac{1}{P^{n-3/2-\mu} |\phi|^{\mu+1}} \qquad \mu \geq 1$$

We will prove the more difficult case *iii*), as cases *i*) and *ii*) follow similar arguments, and we leave the details of those cases to the reader.

We use the same notation as in Theorem 3.10 *iii*). As in Theorem 3.10 *iii*) we divide the estimates into subcases depending on whether $z \in U_\varepsilon$.

Subcase a). Suppose $z \in U_\varepsilon(p_j)$. We estimate

$$\int_{U_{2\varepsilon}(p_j)} \frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta} dV(\zeta) \tag{4.5}$$

and

$$\int_{D_\epsilon \setminus U_{2\varepsilon}} \frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta} dV(\zeta). \tag{4.6}$$

We break up the integral in (4.5) into integrals over $E_c(z)$ and its complement, where c is as in Lemma 3.6, and we choose $c < 1$. Thus, in $E_c(z)$, we have $|\zeta - z| \lesssim \gamma(\zeta)$.

In the case $U_{2\varepsilon}(p_j) \cap E_c(z)$, we use a coordinate system, $s = -r(\zeta)$, $t_1, \ldots, t_{2n-1}$, and estimate

$$\begin{aligned} \int_{U_{2\varepsilon}(p_j)\cap E_c(z)} & \frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta} dV(\zeta) \\ & \lesssim \int_{\mathbb{R}^2_+} \frac{t^{2n-2}}{\gamma(\zeta)\gamma(z)s^\delta(\theta+s+t^2)^{\mu+1}(s+t)^{2n-3-2\mu}} dsdt \\ & \lesssim \int_{\mathbb{R}^2_+} \frac{t^{2\mu-1}}{s^\delta(\theta+s+t^2)^{\mu+1}} dsdt \\ & \lesssim \int_{\mathbb{R}^2_+} \frac{1}{s^\delta(\theta^{1/2}+s^{1/2}+t)^3} dsdt \\ & \lesssim \int_0^\infty \frac{1}{s^\delta(\theta+s)} ds \\ & \lesssim \frac{1}{\theta^\delta}, \end{aligned} \tag{4.7}$$

where we use the notation $\mathbb{R}^j_+ = \overbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}^{j \text{ times}}$.

We now estimate the integral

$$\int_{U_{2\varepsilon}(p_j)\setminus E_c(z)} \frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta} dV(\zeta). \tag{4.8}$$

Recall from above that with the coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ so that around the critical point, p_j we have

$$-r(\zeta) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2$$

and with $w_1, \ldots, w_{2n}$ defined by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1 \le \alpha \le m \\ v_{j_\alpha} & \text{for } m+1 \le \alpha \le 2n, \end{cases}$$

we have $|w(\zeta)| \lesssim |\zeta - z|$ and $|w(\zeta)| \lesssim \gamma(\zeta)$ for $\zeta, z \in U_{2\varepsilon}(p_j)$.

We can therefore bound the integral in (4.8) by

$$\begin{aligned}\int_{U_{2\varepsilon}(p_j)\setminus E_c(z)} & \frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta}dV(\zeta)\\ &\lesssim \int_V \frac{u^{m-1}v^{2n-m-1}}{(u+v)^{2n-2}(\theta+u^2+v^2)(u^2-v^2)^\delta}dudv\\ &\lesssim \int_V \frac{1}{(\theta+u^2)(u^2-v^2)^\delta}dudv, \qquad (4.9)\end{aligned}$$

where V is a bounded region. We make the substitution $v=\tilde{v}u$, since $v^2<u^2$, and write (4.9) as

$$\begin{aligned}\int_0^M \frac{1}{u^{2\delta-1}(\theta+u^2)}du\int_0^1\frac{1}{(1-\tilde{v}^2)^\delta}d\tilde{v} &\lesssim \frac{1}{\theta^\delta}\int_0^M\frac{1}{u^{2\delta-1}(1+u^2)}du\\ &\lesssim \frac{1}{\theta^\delta},\end{aligned}$$

where $M>0$ is some constant. We have therefore bounded (4.5), and we turn now to (4.6).

In $D\setminus U_{2\varepsilon}$ we have that $|\zeta-z|$ and $\gamma(\zeta)$ are bounded from below so

$$\int_{D\setminus U_{2\varepsilon}}\frac{1}{\gamma(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta}dV(\zeta)\lesssim\int_{D\setminus U_{2\varepsilon}}\frac{1}{|r(\zeta)|^\delta}dV(\zeta)\lesssim 1,$$

the last inequality following because in $D\setminus U_{2\varepsilon}$ r can be chosen as a coordinate since $\gamma(\zeta)$ is bounded from below. This finishes subcase a).

Subcase b). Suppose $z\notin U_\varepsilon$. We divide D into the regions $D\cap E_c(z)$ and $D\setminus E_c(z)$.

In $D\cap E_c(z)$ the same coordinates and estimates work here as in establishing the estimates for the integral in (4.7).

In $D\setminus E_c(z)$ we have $|\zeta-z|\gtrsim\gamma(z)$, but $\gamma(z)$ is bounded from below, since $z\notin U_\varepsilon$. We therefore have to estimate

$$\int_D\frac{1}{\gamma(\zeta)|r(\zeta)|^\delta}dV(\zeta),$$

which is easily done by working with the coordinates $w_1,\dots,w_{2n}$ above.

(4.3) is proved similarly. □

Lemma 4.2. *Let T be a tangential vector field and $\varepsilon>0$. For $\epsilon>0$ sufficiently small*

$$\begin{aligned}&i)\ Z_{n+2}:L^2(D)\to L^\infty(D)\\ &ii)\ \|\gamma TZ_4f\|_{C^{1/4-\varepsilon}}\lesssim\|f\|_{L^{\infty,3+\epsilon,0}}\end{aligned}$$

Proof. For i) apply Corollary 3.1 and Theorem 3.12 i), $n+2$ times.

For ii) we let $\alpha<1/4$, and apply the commutator theorem, Theorem 3.4, and consider the two compositions $Z_1\circ Z_1\circ\gamma TA_1\circ Z_1$, and $Z_1\circ Z_1\circ\gamma TE\circ Z_1$. From Theorems 3.10 and 3.12 we can find $\epsilon_1,\dots,\epsilon_4$ such that $0<\epsilon_{j+1}<\epsilon_j$ and such that in the first case we have

$$\begin{aligned}\|Z_1\circ Z_1\circ\gamma TA_1\circ Z_1f\|_{\Lambda_\alpha}&\lesssim\|Z_1\circ\gamma TA_1\circ Z_1f\|_{L^{\infty,\epsilon_1,0}}\lesssim\\ &\|\gamma TA_1\circ Z_1f\|_{L^{\infty,\epsilon_2,\delta}}\lesssim\|Z_1f\|_{L^{\infty,2+\epsilon_3,0}}\lesssim\|f\|_{L^{\infty,3+\epsilon_4,0}}\end{aligned}$$

and, in the second,

$$\|Z_1 \circ Z_1 \circ \gamma TE \circ Z_1 f\|_{\Lambda_\alpha} \lesssim \|Z_1 \circ \gamma TE \circ Z_1 f\|_{L^{\infty,\epsilon_1,0}} \lesssim \|\gamma TE \circ Z_1 f\|_{L^{\infty,1+\epsilon_2,0}} \lesssim \|Z_1 f\|_{\Lambda_{\alpha,3+\epsilon_3}} \lesssim \|f\|_{L^{\infty,3+\epsilon_4,0}},$$

where the second and third inequalities are proved in the same way as Theorem 3.12 *ii*) and *iii*). ☐

We now iterate (4.1) to get

$$\gamma^{3j} f = (Z_1\gamma^{3(j-1)+2} + Z_2\gamma^{3(j-2)+2} + \cdots + Z_j\gamma^2)\bar\partial f + (Z_1\gamma^{3(j-1)+2} + Z_2\gamma^{3(j-2)+2} + \cdots + Z_j\gamma^2)\bar\partial^* f + Z_j f. \tag{4.10}$$

Then we can prove

Theorem 4.3. *For $f \in L^2_{0,q}(D) \cap Dom(\bar\partial) \cap Dom(\bar\partial^*)$, $q \geq 1$, and $\varepsilon > 0$*

$$\|\gamma^{3(n+3)} f\|_{C^{1/4-\varepsilon}} \lesssim \|\gamma^2\bar\partial f\|_\infty + \|\gamma^2\bar\partial^* f\|_\infty + \|f\|_2.$$

Proof. Use Theorems 3.10 *i*) and 3.12 *ii*) and Lemma 4.2 *i*) in (4.10) with $j = n+3$ ☐

We use the notation D^k to denote a k^{th} order differential operator, which is a sum of terms which are composites of k vector fields.

We define

$$Q_k(f) = \sum_{j=0}^{k} \|\gamma^{j+2} D^j \bar\partial f\|_\infty + \sum_{j=0}^{k} \|\gamma^{j+2} D^j \bar\partial^* f\|_\infty + \|f\|_2.$$

T^k will be used for a k-th order tangential differential operator, which is a sum of terms which are composites of k tangential vector fields.

Lemma 4.4. *Let T^k be a tangential operator of order k. For $\varepsilon, \epsilon > 0$*

$$\|\gamma^{3(n+6)+8k+\epsilon} T^k f\|_{C^{1/4-\varepsilon}} \lesssim Q_k(f).$$

Proof. We first prove

$$\|\gamma^{3(n+2)+9+8k+\epsilon} T^k f\|_{L^\infty} \lesssim Q_k(f). \tag{4.11}$$

The proof is by induction in which the first step is proved as was Theorem 4.3. We choose $j = 3$ in (4.10) and then apply (4.10) to $\gamma^{3(n+2)+7k} f$ to get

$$\gamma^{3(n+2)+9+7k} f = Z_1\gamma^2\bar\partial f + Z_1\gamma^2\bar\partial^* f + Z_3\gamma^{3(n+2)+7k} f.$$

We then apply $\gamma^\epsilon(\gamma T)^k$, where T is a tangential operator. We use the commutator theorem, Theorem 3.4, to show

$$\gamma^{3(n+2)+9+8k+\epsilon} T^k f = \gamma^\epsilon \sum_{j=0}^{k-1} Z_3\gamma^{3(n+2)+7k+j} T^j f + \gamma^\epsilon\gamma T Z_3 \gamma^{3(n+2)+8k-1} T^{k-1} f + \gamma^\epsilon \sum_{j=0}^{k} Z_1\gamma^{j+2} T^j \bar\partial f + \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar\partial^* f. \tag{4.12}$$

By Lemma 4.1 and the induction hypothesis, we conclude the L^∞ norm of the first term on the right hand side of (4.12) is bounded by $Q_{k-1}(f)$.

In the same way we proved Lemma 4.2, we have

$$\gamma T Z_3 : L^{\infty,3+\epsilon',0}(D) \to L^{\infty,\epsilon,0}(D),$$

for some $0 < \epsilon' < \epsilon$ and so the L^∞ norm of the second term is bounded by

$$\|\gamma^{3(n+2)+8k+2+\epsilon'} T^{k-1} f\|_{L^\infty} \lesssim \|\gamma^{3(n+2)+9+8(k-1)} T^{k-1} f\|_{L^\infty} \lesssim Q_{k-1}(f).$$

The last two terms on the right side of (4.12) are obviously bounded by $Q_k(f)$, and thus we are done with the proof of (4.11).

To finish the proof of the lemma, we follow the proof of (4.11), and choose $k = 4$ in (4.10), then apply (4.10) to $\gamma^{3(n+2)+7k} f$, and again apply the operators $\gamma^\epsilon (\gamma T)^k$, where T is a tangential operator. In this way, we show

(4.13)

$$\begin{aligned}\gamma^{3(n+2)+12+8k+\epsilon} T^k f =& \gamma^\epsilon \sum_{j=0}^{k-1} Z_4 \gamma^{3(n+2)+7k+j} T^j f + \gamma^\epsilon \gamma T Z_4 \gamma^{3(n+2)+8k-1} T^{k-1} f \\ &+ \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar\partial f + \gamma^\epsilon \sum_{j=0}^{k} Z_1 \gamma^{j+2} T^j \bar\partial^* f.\end{aligned}$$

By Theorems 3.10 *i*) and 3.12 *ii*), for some $\epsilon' > 0$, the first sum on the right hand side of (4.13) has its $C^{1/4-\varepsilon}$ norm bounded by

$$\|Z_3 \gamma^{3(n+2)+7k+\epsilon'+j} T^j f\|_{L^\infty} \lesssim Q_{k-1}(f)$$

from above. We can use Lemma 4.2 *ii*) to show the $C^{1/4-\varepsilon}$ norm of the second term is bounded by

$$\|\gamma^{3(n+2)+10+8(k-1)+\epsilon'} T^{k-1} f\|_\infty \lesssim Q_{k-1}(f)$$

as above.

The last two terms on the right hand side of (4.13) are easily seen to be bounded by $Q_k(f)$, and this finishes Lemma 4.4. □

In order to generalize Lemma 4.4 to include non-tangential operators, we use the familiar argument of utilizing the ellipticity of $\bar\partial \oplus \bar\partial^*$ to express a normal derivative of a component of a $(0,q)$-form, f, in terms of tangential operators acting on components of f and components of $\bar\partial f$ and $\bar\partial^* f$. With the $(0,q)$-form f written

$$f = \sum_{|J|=q} f_J \bar\omega^J$$

locally, we have the decomposition in the following form:

$$\gamma N f_J = \sum_{jK} a_{JjK} \gamma T_j f_K + \sum_L b_{JL} f_L + \sum_M c_{JM} \gamma (\bar\partial f)_M + \sum_P d_{JP} \gamma (\bar\partial^* f)_P, \tag{4.14}$$

where $N = L_n + \bar L_n$ is the normal vector field, and $T_1, \dots, T_{2n-1}$ are the tangential fields as described in Section 3. The coefficients a_{JjK}, b_{JL}, c_{JM}, and d_{JP} are all of the form $\mathcal{E}_{0,0}$ and the index sets are strictly ordered with $J, K, L, M, P \subset \{1, \dots, n\}$, $|J| = |K| = |L| = q$, $|M| = q+1$, $|P| = q-1$, $j = 1, \dots, 2n-1$. The decomposition is well known in the smooth case (see [3]) and to verify (4.14) in a neighborhood of $\gamma = 0$, one may use the coordinates $u_{j_1}, \dots, u_{j_m}, v_{j_{m+1}}, \dots, v_{j_{2n}}$ as in (3.11) above. For instance, integrating by parts to compute $\bar\partial^* f$ leads to terms of the form $\mathcal{E}_{0,-1} f_J$, whereby multiplication by γ allows us to absorb these terms into b_{JL}.

It is then straightforward how to generalize Lemma 4.4. Suppose D^k is a k^{th} order differential operator which contains the normal field at least once. In $\gamma^k D^k$ we commute γN with terms of the form γT, where T is tangential, and we consider the operator $D^k = D^{k-1} \circ \gamma N$, where D^{k-1} is of order $k-1$. The error terms due to the commutation involve differential operators of order $\leq k-1$. From (4.14) we just have to consider $D^{k-1}\gamma T f$, $D^{k-1}\bar{\partial} f$, and $D^{k-1}\bar{\partial}^* f$. The last two terms are bounded by $Q_{k-1}(f)$, and we repeat the process with $D^{k-1}\gamma T f$, until we are left with k tangential operators for which we can apply Lemma 4.4.

We thus obtain the weighted C^k estimates

Theorem 4.5. *Let $f \in L^2_{0,q}(D) \cap Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$, $q \geq 1$, $\alpha < 1/4$, and $\epsilon > 0$. Then*

$$\|\gamma^{3(n+6)+8k+\epsilon} f\|_{C^{k+\alpha}} \lesssim Q_k(f).$$

As an immediate consequence we obtain weighted C^k estimates for the canonical solution to the $\bar{\partial}$-equation.

Corollary 4.6. *Let $q \geq 2$ and let N_q denote the $\bar{\partial}$-Neumann operator for $(0,q)$-forms. Let f be a $\bar{\partial}$-closed $(0,q)$-form. Then for $\alpha < 1/4$ and $\epsilon > 0$, the canonical solution, $u = \bar{\partial}^* N_q f$ to $\bar{\partial} u = f$, satisfies*

$$\|\gamma^{3(n+6)+8k+\epsilon} u\|_{C^{k+\alpha}} \lesssim \|\gamma^{k+2} f\|_{C^k} + \|f\|_2.$$

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